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Determination of Two-Loop $\pi\pi$ Scattering Amplitude Parameters

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Abstract

The chiral expansion of the $\pi\pi$ amplitude to the order of two loops was expressed in terms of six independent parameters in a previous paper: four of these are shown here to satisfy sum rules. Their derivation, where crossing symmetry plays a key role, is explained. Their convergence properties are studied and their practical evaluation, in terms of the available data on $\pi\pi$ phase shifts above 0.5 GeV, is discussed. Below 0.5 GeV, the chiral amplitude itself is employed, such that the parameters are determined in a self-consistent way. Some care is devoted to the estimate of the errors.

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1. Introduction

In a previous paper [1] (referred to as I in the sequel), we showed that the $\pi\pi$ elastic scattering amplitude to the order of two loops in the chiral expansion takes the form of an analytic expression that depends on six parameters ($\alpha,\ \beta,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$) which are not determined by chiral symmetry. In the present work, the possibility of determining these parameters from experiment will be discussed. It will be shown that four of them, $\lambda_1, ..., \lambda_4$, can be inferred with rather good accuracy from existing $\pi\pi$ scattering data at energies $\sqrt{s} \geq 500$ MeV. This determination is merely based on very general properties of the scattering amplitude and it is completely model independent. Before entering into the details of the subject [2] a few comments on the low-energy constants characterizing higher orders in the low-energy expansion may be useful.

Upon extending the program of the chiral perturbation theory (χPT) beyond one-loop order [2, 3], one faces the problem of the proliferation of low-energy constants: at order $O(p^6)$ there are more than a hundred new unknown parameters [4]. At one-loop level, the predictive power of χPT resides in the fact that the same (combinations of) low-energy constants often appear in different observables. This fact, of course, reflects chiral Ward identities and unitarity and it is encoded in the effective lagrangian. However, at two-loop order, eliminating unknown parameters in the same way by comparing different observables becomes problematic in practice, though not impossible in principle. In spite of this difficulty, successful two-loop calculations do exist [5], in which the influence of the new $O(p^6)$ constants remains rather limited, at least in a particular kinematical region.

In general, it seems plausible that at $O(p^6)$ accuracy, each experimentally relevant process will be described by its own, specific set of low-energy constants, which have to be determined from considerations operating beyond the strict framework of the low-energy expansion. Models aiming at such a determination have been proposed, and already applied in the past both at the $O(p^4)$ [6, 7] and $O(p^6)$ [5] levels. They incorporate low-lying meson resonances into the effective lagrangian, thereby extending its domain of applicability. The unknown low-energy constants are then obtained by fitting the resonance parameters and couplings first, and then integrating out the resonance degrees of freedom (expanding in inverse powers of their masses). While this procedure is essentially unique and presumably rather accurate in the case of vector or axial-vector resonances [7], it is less convincing as far as scalar exchanges are concerned: i) first, incorporating spin 0 resonances into $\mathcal{L}_{eff}$ in a way consistent with chiral symmetry is not a unique procedure. For obvious kinematical reasons, the criterion of matching with QCD at short distances [7] is less stringent than in the case of spin-1 exchanges. Too many terms survive this criterion, which even requires the presence of scalar couplings with a non-minimal number of derivatives [8] (which were not considered in [6]). ii) Next, it has been suggested that the spectrum and couplings of the scalar mesons are subject to particularly strong distortions due

[2] A brief account of the results of the present paper has already been presented in I (sec. 4).

to unitarity and final-state interactions [9]. It may be misleading to describe the 0^{++} channels by a set of quasi-real poles or, equivalently, to identify the resonance masses and couplings in the extended effective lagrangian with the corresponding entries in the PDG tables [10]. For all these reasons, a more realistic and less model dependent method of determination of low-energy constants is needed, especially in the symmetry breaking sector which is particularly sensitive to scalar exchanges.

In this paper, such an unambiguous method is worked out in the case of elastic $\pi\pi$ scattering. It may be viewed as a generalization of the method of chiral sum rules for various two-point functions [11, 12], which has been already used to determine some $O(p^6)$ constants in connection with the reaction $\gamma\gamma \to \pi^0\pi^0$ [12]. The method makes use of analyticity, unitarity and crossing symmetry to relate the existing experimental information on the $\pi\pi$ scattering amplitude at medium energies, $0.5\ \mathrm{GeV} \leq \sqrt{s} \leq 1.9\ \mathrm{GeV}$, to its behaviour in the threshold region, where χPT applies. The comparison yields six sum rules for the six unknown parameters $\alpha,\ \beta,\ \lambda_1, ..., \lambda_4$ which define the perturbative $\pi\pi$ amplitude up to two loops. Four of these sum rules converge rapidly enough to be practically insensitive to the poorly known high-energy part of the $\pi\pi$ amplitude. In the absence of information on this part, the remaining two sum rules are of little use in practice. For this reason, our method merely serves to fix the constants $\lambda_1, ..., \lambda_4$, leaving the parameters α and β undetermined. This, of course, is reminiscent of the derivation of the Roy equations [13, 14] for the $\pi\pi$ scattering amplitude: here, as well, after imposing crossing symmetry and fixed-t analyticity, one remains with two arbitrary subtraction constants [13, 14]. The parallel with the Roy equations will be further developed in sec. 2, where the derivation of the sum rules is presented in details.

The $\pi\pi$ amplitude is a rare example of a low-energy observable which is sensitive to the mechanism of spontaneous chiral symmetry breaking, in particular to the strength of quark antiquark condensation in the QCD vacuum [15, 16]. Indeed, elastic $\pi\pi$ scattering provides the best way of testing the standard postulate, according to which the quark condensate

$$B_0 = -\lim_{m_q \to 0} \frac{1}{F_\pi^2} < \bar{q}q > \tag{1}$$

is large enough (c.f. $B_0 \sim 1.5$ GeV) to dominate the response to the perturbation by light quark masses [17]. This hypothesis is at the basis of the standard χPT [2, 3], which explicitly *assumes* that the GOR relation $2\hat{m}B_0 = M_\pi^2$ is not violated by more than a few percent [2]. This assumption has not yet been tested experimentally, and it is not a necessary prerequisite for a consistent chiral expansion: indeed, it was shown in [15, 16, 12] that a systematic, more general expansion ($G\chi$PT) can be developed, which neither requires B_0 to be large, nor the expansion of M_π^2 to be dominated by $2\hat{m}B_0$ (see [18] for a recent review). The two-loop expression of the $\pi\pi$ amplitude displayed in I holds in full generality, independently of any particular mechanism of chiral symmetry breaking. The strength of quark condensation merely affects the values of the parameters $\alpha,\ \beta,\ \lambda_1, ..., \lambda_4$, but not the analytic form of the amplitude. The standard and the generalized χPT differ in the expansion of these parameters in terms of quark masses, chiral

logarithms and the renormalized (quark mass independent) constants of $\mathcal{L}_{eff}$. This expansion was worked out very recently [19] up to chiral order six in the standard χPT. This kind of information is particularly useful for the two parameters α and β, the expansion of which start at the lowest $O(p^2)$ order of the χPT, and higher order contributions are expected to remain relatively small. In the standard χPT, both α and β are predicted to be close to 1 [2] (within $10-20\%$, say), while in the alternative situation of a small condensate the value of α would be $2-3$ times larger, depending on the actual value of the quark mass ratio $m_s/\hat{m}$, which remains a free parameter in the $G\chi$PT. In all cases the deviation of β from 1 should remain small: for more details, see I. In contrast, the expansions of the parameters $\lambda_1,\ \lambda_2$ and $\lambda_3,\ \lambda_4$ start at order $O(p^4)$ and $O(p^6)$, respectively. Consequently, these parameters are more sensitive to the unknown $O(p^6)$ constants of $\mathcal{L}_{eff}$ and their values are thus expected to be more difficult to estimate on the basis of a χPT calculation alone. It is gratifying that the experimental $\pi\pi$ data at medium energies allow to determine $\lambda_1,...,\lambda_4$, and it is significant that the same data leave α and β undetermined. Our analysis further suggests that the values of $\lambda_1,...,\lambda_4$ are practically insensitive to the strength of quark condensation. We checked that by allowing α to vary in a rather wide range (from 1 to 4), the calculated values of the λ_i's were only affected at the level of a few percent. In particular, the values we obtain should be used within the framework of the standard χPT [19] (see sec. 3.5).

In order to disentangle the large and small condensate alternatives, it would be ideal to be able to exploit the variation of the parameters $\alpha,\ \beta,\ \lambda_1,...,\lambda_4$ as functions of quark masses and chiral logarithms as predicted in the standard and generalized χPT, respectively. Unfortunately, this kind of theoretical information is not suitable for a direct experimental test, since Nature has already made its choice of quark masses. On the other hand, the forthcoming high-precision data near threshold [20, 21] should allow to measure the parameter α, for which the alternatives in question differ by a factor $2-3$. Obviously, determining all six parameters of the two-loop amplitude from a fit to the low-energy data appears practically hopeless. For this reason, the model independent determination of the parameters $\lambda_1,...,\lambda_4$ reported in this paper becomes of a fundamental importance. Special care is devoted to the discussion of the experimental inputs and of the uncertainties in the results (sec. 3). A reliable estimate of the errors is indeed crucial to correctly assess the size of the expected theoretical uncertainty in future experimental determinations of the critical parameter α.

2. Derivation of the sum rules

We start by giving a brief description of the $\pi\pi$ amplitude to two loops, in order to set up the definition of the parameters for which we intend to establish the sum rules. We will neither repeat the derivation nor even give the complete formulas, all these details can be found in I. The T-matrix element for the process $\pi^a\pi^b \to \pi^c\pi^d$ is expressed in a standard way in terms of a single function $A(s|t,u)$, symmetric in t,u, and where the Mandelstam variables s,t,u satisfy

$s+t+u=4M_\pi^2$, as

$$T^{ab,cd}(s,t,u)=A(s|t,u)\delta^{ab}\delta^{cd}+A(t|s,u)\delta^{ac}\delta^{bd}+A(u|t,s)\delta^{ad}\delta^{bc}. \tag{2}$$

At two-loop order, the function $A(s|t,u)$ may be written as a sum of two terms

$$A^{2-loop}(s|t,u)=A^{pol}(s|t,u)+A^{cut}(s|t,u)\ , \tag{3}$$

where the former is a polynomial in the Mandelstam variables and the latter collects the cuts of the amplitude. The polynomial is of the third order and satisfies crossing symmetry. It may be parametrized as

$$\begin{aligned} A^{pol}(s|t,u)= \quad & \frac{\beta}{F_\pi^2}\left(s-\frac{4M_\pi^2}{3}\right)+\frac{\alpha}{F_\pi^2}\frac{M_\pi^2}{3} \\ & +\frac{\lambda_1}{F_\pi^4}(s-2M_\pi^2)^2+\frac{\lambda_2}{F_\pi^4}\left[(t-2M_\pi^2)^2+(u-2M_\pi^2)^2\right] \\ & +\frac{\lambda_3}{F_\pi^6}(s-2M_\pi^2)^3+\frac{\lambda_4}{F_\pi^6}\left[(t-2M_\pi^2)^3+(u-2M_\pi^2)^3\right]\ , \end{aligned} \tag{4}$$

which displays the six parameters $\alpha,\ \beta,\ \lambda_1,\ \lambda_2,\ \lambda_2,\ \lambda_4$ of the two-loop amplitude. The part containing the cuts in the complex plane can be expressed in terms of three functions of a single variable $\hat{W}_a(z),\ a=0,1,2$, which are i) analytic in the whole complex plane in the variable z except for a right-hand cut on the real axis $[4M_\pi^2,\infty]$, and ii) asymptotically bounded when $z\to\infty$, such that $\hat{W}_a/z^4\to 0$ for $a=0,2$ and $\hat{W}_1/z^3\to 0$. In terms of these functions, $A^{cut}(s|t,u)$ can be written as

$$\begin{aligned} \frac{1}{32\pi}A^{cut}(s|t,u) &= \frac{1}{3}\left[\hat{W}_0(s)-\hat{W}_2(s)\right] \\ &+ \frac{1}{2}\left[3(s-u)\hat{W}_1(t)+\hat{W}_2(t)\right] \\ &+ \frac{1}{2}\left[3(s-t)\hat{W}_1(u)+\hat{W}_2(u)\right]\ . \end{aligned} \tag{5}$$

The fact that the amplitude to two loops has the general structure described in the above equations (3),(4),(5) was first proved in [16]. In the subsequent paper I, the explicit form of the three functions $\hat{W}_a(z)$ was obtained [3]

$$\hat{W}_a(z)=\sum_{n=0}^{4}w_a^n(z)\bar{K}_n(z)\ , \tag{6}$$

where

$$\bar{K}_0(z)=\frac{1}{16\pi^2}\left\{2+\sqrt{1-4M_\pi^2/z}\ \ln\left(1-\frac{2}{1+\sqrt{1-4M_\pi^2/z}}\right)\right\} \tag{7}$$

[3] The functions $W_a(z)$ introduced in I differ from the functions $\hat{W}_a(z)$ given in (6) by polynomials which are defined up to an ambiguity described in appendix B of [16].

is the Chew-Mandelstam function which is already present at one-loop order, and the other four functions $\bar{K}_1, ..., \bar{K}_4$ are simple combinations of $\bar{K}_0$, $(\bar{K}_0)^2$ and $(\bar{K}_0)^3$. The set of functions $w_a^n(z)$ are third-degree polynomials in z which are tabulated in I. These polynomials depend on four constants, $\alpha,\ \beta,\ \lambda_1,\ \lambda_2$ which are the same (up to terms of order $O(p^8)$ in the amplitude) as those which appear in (4). The dependence is cubic in α and β and linear in λ_1 and λ_2. Finally, we note that there is a simple relation among the discontinuities of the functions $\hat{W}_a$ and the discontinuities of the S and P partial waves along the right-hand cut, $s \geq 4M_\pi^2$:

$$\mathrm{Im} f_0^a(s) = \mathrm{Im}\hat{W}_a(s),\ a = 0, 2, \qquad \mathrm{Im} f_1^1(s) = (s - 4M_\pi^2)\,\mathrm{Im}\hat{W}_1(s). \tag{8}$$

This relation is actually used in the construction of the functions $\hat{W}_a$ in a recursive way, starting from the expression of the partial waves at order $O(p^2)$ and using unitarity to generate the imaginary part at a higher order. This technique was used for the first time in [22] to obtain the one-loop $\pi\pi$ amplitude in the case $M_\pi = 0$.

2.1 Roy dispersion relations

Due to the Froissart bound, the $\pi\pi$ scattering amplitude obeys fixed t, twice-subtracted dispersion relations which involve three subtraction functions. It was shown by Roy [13] and by Basdevant *et al.* [14] that these three functions are entirely determined, up to two constants (which, for instance, may be taken as the two scattering lengths a_0^0 and a_0^2), once crossing symmetry is imposed. The key step in the derivation of the sum rules is to equate, in the low energy region, the perturbative expansion of the amplitude (3) (4) (5) with the Roy dispersive representation. We start by rederiving this representation in a form which will be convenient for our purpose.

Consider the s-channel isospin $I = 0, 1, 2$ amplitudes $F^I(s,t)$, defined as

$$\begin{aligned} F^0(s,t) &= \frac{1}{32\pi}\left\{3A(s|t,u) + A(t|s,u) + A(u|s,t)\right\}\ , \\ F^1(s,t) &= \frac{1}{32\pi}\left\{A(t|s,u) - A(u|s,t)\right\}\ , \\ F^2(s,t) &= \frac{1}{32\pi}\left\{A(t|s,u) + A(u|s,t)\right\}\ , \end{aligned} \tag{9}$$

and form a 3-vector

$$\mathbf{F}(s,t) = \begin{pmatrix} F^0(s,t) \\ F^1(s,t) \\ F^2(s,t) \end{pmatrix}. \tag{10}$$

The Froissart bound allows one to write a fixed t dispersion relation with two subtractions

$$\mathbf{F}(s,t) = C_{st}[\mathbf{a}_+(t) + (s-u)\mathbf{a}_-(t)] + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty}\frac{dx}{x^2}\left(\frac{s^2}{x-s} + \frac{u^2 C_{su}}{x-u}\right)\mathrm{Im}\mathbf{F}(x,t)\ , \tag{11}$$

where C_{st}, C_{su} and C_{tu} are the $s-t$, $s-u$ and $t-u$ crossing matrices

$$C_{st} = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}, \quad C_{su} = \begin{pmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{pmatrix}, \quad C_{tu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad (12)$$

We have also introduced the vectors

$$\mathbf{a}_+(t) = \begin{pmatrix} a_0(t) \\ 0 \\ a_2(t) \end{pmatrix}, \qquad \mathbf{a}_-(t) = \begin{pmatrix} 0 \\ a_1(t) \\ 0 \end{pmatrix}, \quad (13)$$

which collect three arbitrary functions of t. As was observed in [13, 14], the constraint of crossing symmetry reduces this arbitrariness down to two real constants. From (11), $\mathbf{F}$ manifestly satisfies $s-u$ crossing by construction, *i.e.* $\mathbf{F}(s,t) = C_{su}\mathbf{F}(u,t)$. One must impose $s-t$ crossing symmetry, and then $t-u$ crossing symmetry will follow automatically.

It proves convenient to split the integration range into two parts, $[4M_\pi^2, E^2]$ and $[E^2, \infty]$. In the classic study of the Roy equations [23] E was taken to be rather large, $E \simeq 1.5$ GeV. Here, for the purpose of generating a form which can easily be compared with the two-loop expression, we will take E to be sufficiently small, so that in the range $4M_\pi^2 \le x \le E^2$ the contributions of the partial waves with $\ell \ge 2$ to the imaginary part of the amplitude are negligibly small as compared to the S- and P-wave contributions. This condition is satisfied for $E \lesssim 1$ GeV. For $x \le E^2$ we can write

$$\mathrm{Im}\,\mathbf{F}(x,t) = \begin{pmatrix} \mathrm{Im}\, f_0^0(x) \\ 0 \\ \mathrm{Im}\, f_0^2(x) \end{pmatrix} + 3\Big(1 + \frac{2t}{x-4M_\pi^2}\Big) \begin{pmatrix} 0 \\ \mathrm{Im}\, f_1^1(x) \\ 0 \end{pmatrix} . \quad (14)$$

Inserting this into the dispersion relation (11), there will appear three functions $W_a^E(z)$ which are analogous to the functions $\hat{W}_a(z)$ in the two-loop expression:

$$W_a^E(z) = \frac{z^2}{\pi} \int_{4M_\pi^2}^{E^2} \frac{dx}{x^2} \frac{\mathrm{Im}\, f_0^a(x)}{x-z} \quad (a=0,2) , \qquad W_1^E(z) = \frac{z}{\pi} \int_{4M_\pi^2}^{E^2} \frac{dx}{x(x-4M_\pi^2)} \frac{\mathrm{Im}\, f_1^1(x)}{x-z} . \quad (15)$$

We collect these functions into vectors

$$\mathbf{W}_+^E(z) = \begin{pmatrix} W_0^E(z) \\ 0 \\ W_2^E(z) \end{pmatrix}, \qquad \mathbf{W}_-^E(z) = \begin{pmatrix} 0 \\ W_1^E(z) \\ 0 \end{pmatrix}. \quad (16)$$

The amplitude $\mathbf{F}(s,t)$ can then be expressed as a sum of two terms

$$\mathbf{F}(s,t) = \mathbf{W}^E(s,t) + \mathbf{Z}^E(s,t) . \quad (17)$$

The first term is constructed from the integrals W_a^E in order to satisfy crossing symmetry exactly:

$$\begin{aligned}\mathbf{W}^E(s,t) = \quad & \mathbf{W}_+^E(s) + 3(t-u)\mathbf{W}_-^E(s) + C_{su}[\mathbf{W}_+^E(u) + 3(t-s)\mathbf{W}_-^E(u)] \\ & + C_{st}[\mathbf{W}_+^E(t) + 3(s-u)\mathbf{W}_-^E(t)] \ . \end{aligned} \tag{18}$$

The second term, $\mathbf{Z}^E(s,t)$, collects what is left from the original dispersive representation. Remarkably, it can be expressed solely in terms of integrals over the high-energy range $[E^2,\infty]$ and of three arbitrary functions of t. This is achieved by redefining the original subtraction functions $\mathbf{a}_+(t)$ and $\mathbf{a}_-(t)$ into new ones $\mathbf{b}_+(t)$ and $\mathbf{b}_-(t)$ (which are equally arbitrary for the moment), collecting into them as many terms as possible:

$$\begin{aligned}\mathbf{b}_+(t) &= \mathbf{a}_+(t) - \mathbf{W}_+^E(t) + 3(t-4M_\pi^2)(t-2M_\pi^2)(1+C_{tu})\frac{1}{\pi}\int_{4M_\pi^2}^{E^2}\frac{dx}{x^2}C_{st}\mathrm{Im}\,\mathbf{W}_-^E(x) \\ b_1(t) &= a_1(t) - 3W_1^E(t) - 3(t-2M_\pi^2)\frac{1}{\pi}\int_{4M_\pi^2}^{E^2}\frac{dx}{x^2(x+t-4M_\pi^2)}\mathrm{Im}\,f_1^1(x) \\ & \quad -(t-4M_\pi^2)\frac{1}{\pi}\int_{E^2}^{\infty}\frac{dx}{x^2(x-4M_\pi^2)}\mathrm{Im}\,F^{(I_t=1)}(x,t) \ . \end{aligned} \tag{19}$$

We have introduced the amplitudes with given isospin in the t-channel which are defined, as usual, by

$$F^{(I_t=a)}(s,t) = \sum_{I=0}^{2}(C_{st})_{aI}F^I(s,t) \ . \tag{20}$$

These explicit relations between the old and the new subtraction functions will only be useful at a later stage for the derivation of slowly convergent sum rules for the parameters α and β. $\mathbf{Z}^E(s,t)$ has now the following expression:

$$\mathbf{Z}^E(s,t) = C_{st}\Big\{\mathbf{b}_+(t) + (s-u)\mathbf{b}_-(t) + us[\mathbf{h}_+(t,us) + (s-u)\mathbf{h}_-(t,us)]\Big\} \ , \tag{21}$$

where the vectors $\mathbf{h}_+(t,us)$ and $\mathbf{h}_-(t,us)$ are formed in analogy to $\mathbf{a}_\pm$, $\mathbf{W}_\pm$ (see (13), (16)) from the three functions

$$h_a(t,us) = \frac{-1}{\pi}\int_{E^2}^{\infty}\frac{dx\,(2x+t-4M_\pi^2)\,\mathrm{Im}\,F^{(I_t=a)}(x,t)}{x(x+t-4M_\pi^2)[x^2+x(t-4M_\pi^2)+us]} \ , \ a=0,2 \tag{22}$$

and

$$h_1(t,us) = \frac{-1}{\pi}\int_{E^2}^{\infty}\frac{dx\,\mathrm{Im}\,F^{(I_t=1)}(x,t)}{x(x+t-4M_\pi^2)[x^2+x(t-4M_\pi^2)+us]} \ . \tag{23}$$

As announced, these dispersive integrals extend over the high energy range $[E^2,\infty]$.

Up to this point, E was bounded only from above by the requirement that the contribution from the D-waves can be neglected for energies smaller than E. We can now also take E to be sufficiently large compared to twice the pion mass. In other words we require

$$4M_\pi^2 << E^2 < 1\ \mathrm{GeV}^2 \ . \tag{24}$$

Then, in the low energy region, the ratios $s/E^2,\ t/E^2,\ u/E^2$ will be of order $O(p^2)$ in the chiral counting; consequently, we can expand the integrals $h_a(t,us)$ in powers of these quantities and drop the terms which are of chiral order $O(p^8)$ or higher in the amplitude (this is legitimate since we intend to equate this form of the amplitude with the expansion to two-loop chiral order). Hence, we can replace in Eq.(21)

$$h_a(t,us) = h_a(0,0) + t\partial_t h_a(t,0)|_{t=0} + O(p^4)\ (a=0,2), \quad h_1(t,us) = h_1(0,0) + O(p^2)\ . \tag{25}$$

Finally, we must impose $s-t$ crossing symmetry (to the same chiral order) on $\mathbf{Z}^E(s,t)$, *i.e.*

$$\mathbf{Z}^E(t,s) = C_{st}\,\mathbf{Z}^E(s,t) + O(p^8)\ . \tag{26}$$

The standard trick of setting $s=0$ in (26) shows that the subtraction functions $b_a(t)$ must be polynomials once the integrals $h_a(t,us)$ have been expanded (25). These polynomials are entirely determined up to two constants, a and b:

$$\begin{aligned}
b_0(t) &= \frac{M_\pi^2}{3}(5a-8b) + 2tb - \frac{1}{3}t(t-4M_\pi^2)\left[h_0 + 6(t-2M_\pi^2)h_1 + 5h_2\right] \\
b_1(t) &= b - \frac{t}{6}\left[2h_0 + 6(t-2M_\pi^2)h_1 - 5h_2\right] \\
b_2(t) &= \frac{2M_\pi^2}{3}(a+2b) - tb - \frac{t}{6}(t-4M_\pi^2)\left[2h_0 - 6(t-2M_\pi^2)h_1 + h_2\right]\ ,
\end{aligned} \tag{27}$$

where we have introduced the notation

$$h_a \equiv h_a(0,0), \qquad h_a' \equiv \partial_t h_a(t,0)|_{t=0}\ . \tag{28}$$

The crossing relation (26) is then identically satisfied provided the following combination of high energy integrals vanishes in the chiral limit:

$$2h_0' - 5h_2' - 9h_1 = O(p^2)\ . \tag{29}$$

(Using resonance saturation, this constraint may be converted into an amusing relation between the masses and the widths of the $f_2(1270)$ and the $\rho_3(1690)$ resonances [4].) This completes the construction of a dispersive representation of the amplitude $\mathbf{F}(s,t)$ in terms of two arbitrary parameters, a and b. Explicitly, from (18) and (27) one obtains the following expression for the corresponding function $A(s|t,u)$

$$\begin{aligned}
\tfrac{1}{32\pi}\,A^{disp}(s|t,u) = {} & \tfrac{1}{3}\left[W_0^E(s) - W_2^E(s)\right] + \tfrac{3}{2}(s-u)W_1^E(t) + \tfrac{1}{2}W_2^E(t) \\
& + \tfrac{3}{2}(s-t)W_1^E(u) + \tfrac{1}{2}W_2^E(u) \\
& + h_1(-s^3 + 6M_\pi^2 s^2 - 8M_\pi^4 s) + \tfrac{1}{3}(h_0' - h_2')stu - \tfrac{1}{2}h_2 s(s-4M_\pi^2) + \tfrac{1}{3}(h_0-h_2)tu \\
& + bs + \tfrac{1}{3}(a-4b)M_\pi^2 \quad + O(p^8)\ .
\end{aligned} \tag{30}$$

[4] This relation reads: $63\Gamma_\rho M_f^4 = 5\Gamma_f M_\rho^4$, where M_f and M_ρ are the masses of the f_2 and the ρ_3 mesons and $\Gamma_f,\ \Gamma_\rho$ are the respective $\pi\pi$ partial widths.

Expression (30) represents the final form of the dispersive $\pi\pi$ amplitude. We recall that the functions W_a^E are computed from the imaginary parts of the partial-wave amplitudes $f_0^0(s)$, $f_1^1(s)$ and $f_0^2(s)$ (see (15)) in the domain $4M_\pi^2 \leq s \leq E^2$, *i.e.* in the region where experimental information is incomplete, while the constants h_a and h'_a (28) can be evaluated using existing experimental data at higher energy: the part involving these functions corresponds to the so-called driving terms in the Roy equations [23]. Indeed, from $A^{disp}(s|t,u)$ one can compute the real parts of f_0^0, f_1^1 and f_0^2, which are thus given in terms of the corresponding imaginary parts and the driving terms. These integral equations, together with the elastic unitarity relations (which hold to good accuracy below the $K\bar{K}$ threshold) constitute the Roy equations. Numerical solutions to these equations were constructed in [23][24] taking as the two free parameters the scattering lengths a_0^0 and a_0^2. Constraining these solutions to match with the experimental data in the $600 - 800$ MeV region, it was shown that a_0^0 is left essentially undetermined but that a_0^2 gets strongly correlated with a_0^0 (this relationship is known as the Morgan-Shaw band in the literature [25]). We are going to show that in a large domain of parameters α and β, the perturbative amplitude $A^{2-loop}(s|t,u)$ of Eq.(3) provides a rather accurate representation of the numerical solutions of the Roy equations (up to energies $\sqrt{s} \approx 0.5$ GeV), provided the values of the parameters λ_1, λ_2, λ_3 and λ_4 are properly chosen.

2.2 Equating the dispersive and perturbative amplitudes

Let us now restrict ourselves to the domain where the Mandelstam variables $s,\ t,\ u$ are very small (compared to 1 GeV^2). In this domain, $A^{disp}(s|t,u)$ must be identical with $A^{2-loop}(s|t,u)$, except for terms of chiral order $O(p^8)$ or higher. Let us then subtract the part which contains all the cuts of the two-loop expression, $A^{cut}(s|t,u)$ (see (5)) from the dispersive representation (30) of $A(s|t,u)$: this difference must be identical to the polynomial part of the chiral amplitude, $A^{pol}(s|t,u)$ (c.f. Eq.(4)). Indeed, at low energies, the discontinuity in the differences

$$\mathrm{Im}\left[W_a^E(z) - \hat{W}_a(z)\right] = \mathrm{Im}\, f_0^a(z) - \mathrm{Im}\, f_0^a(z)|_{2-loop} \qquad (a = 0, 2) \tag{31}$$

is, by definition, a quantity of chiral order eight. A similar relation also holds for the isospin one case. In the range where $z << 1$ GeV^2, these discontinuities can be neglected and the differences $W_a^E(z) - \hat{W}_a(z)$ are analytic functions which are well-approximated by polynomials

$$32\pi[W_a^E(z) - \hat{W}_a(z)] = \sum_{k=0}^{3} I_a^k z^k + O(z^4)\ . \tag{32}$$

Inserting these expansions into the difference $A^{disp}(s|t,u) - A^{2-loop}(s|t,u)$, we generate six sum rules by recognizing that the polynomial obtained in this way should vanish up to terms of order $O(p^8)$. It is then a simple exercise to derive:

$$\frac{\lambda_1}{F_\pi^4} = \frac{1}{3}(I_0^2 - I_2^2) - 3I_1^1 + 2M_\pi^2(I_0^3 - I_2^3 - 3I_1^2) + \frac{16\pi}{3}(h_0 - 4h_2)$$

$$\frac{\lambda_2}{F_\pi^4} = \frac{1}{2}I_2^2 + \frac{3}{2}I_1^1 + 3M_\pi^2(I_2^3 + I_1^2) - \frac{16\pi}{3}(h_0 - h_2)$$
$$\frac{\lambda_3}{F_\pi^6} = \frac{1}{3}(I_0^3 - I_2^3) + I_1^2 + \frac{32\pi}{9}(h_0' - h_2' - 9h_1) \tag{33}$$
$$\frac{\lambda_4}{F_\pi^6} = \frac{1}{2}(I_2^3 - I_1^2) + \frac{32\pi}{9}(h_0' - h_2') \ ,$$

where the entries I_a^k are defined in (32).

The remaining two equations similar to (33) concern the parameters α and β. However, these depend explicitly on the two subtraction constants a and b of the Roy representation (30):

$$\frac{\alpha}{F_\pi^2} = \frac{4}{3}\left[I_0^1 + 2I_2^1 + M_\pi^2(I_0^2 + 2I_2^2)\right] + 32\pi\left[2M_\pi^2(h_0 + 2h_2) + \frac{8}{3}M_\pi^4(h_0' - h_2' - 3h_1) + a\right]$$
$$\frac{\beta}{F_\pi^2} = \frac{1}{3}I_0^1 - \frac{5}{6}I_2^1 + 4M_\pi^2\left(\frac{1}{3}I_0^2 - \frac{5}{6}I_2^2\right) + 12M_\pi^4\left(\frac{1}{3}I_0^3 - \frac{5}{6}I_2^3 - \frac{1}{2}I_1^2\right) + 32\pi\left[4M_\pi^4 h_1 + b\right] \ . \tag{34}$$

Consequently, one cannot, *a priori*, make use of these equations to determine α and β.

The sum rules (33) and (34) represent the minimal set of necessary and sufficient conditions ensuring the compatibility of the dispersive representation (30) with the chiral expansion of the amplitude up to and including the two-loop accuracy. It is worth stressing the role of crossing symmetry in the derivation of these sum rules.

2.3 Barely converging sum rules for the parameters α and β

The convergence of the high-energy integrals in Eqs. (33) is guaranteed by the Froissart bound. Actually, as will be seen in sec. 3, the sum rules (33) are rapidly convergent and rather independent of the details of high-energy asymptotics. Additional information on the high-energy behaviour of the amplitude would yield additional sum rules. The standard Regge pole phenomenology, for instance, suggests that the $I_t = 1$ t-channel isospin amplitude $F^{(I_t=1)}(s,t)$ behaves asymptotically as $s^{\alpha_\rho(t)}$ where $\alpha_\rho(t)$ is the ρ trajectory with the intercept $\alpha_\rho(0) \simeq 1/2$. According to this picture, the amplitude $F^{(I_t=1)}$ satisfies a once-subtracted dispersion relation. This idea was used in the past to generate a sum rule for the P-wave scattering length a_1^1 and for the combination $2a_0^0 - 5a_0^2$ [26]. Comparing the once-subtracted with the twice-subtracted fixed t dispersion relation (11), one obtains the following expression for the subtraction function $a_1(t)$:

$$a_1(t) = \frac{1}{\pi}\int_{4M_\pi^2}^\infty \frac{dx}{x^2}\,\mathrm{Im}\,F^{(I_t=1)}(x,t) \ . \tag{35}$$

Similarly, the amplitude with $F^{(I_t=2)}(s,t)$ is dominated by exotic t-channel exchanges and it is expected to be asymtotically suppressed. It is even conceivable that it satisfies an unsubtracted

dispersion relation [25]. Under this assumption, one obtains the following sum rule for the subtraction function $a_2(t)$ in (11):

$$a_2(t) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty}\frac{dx}{x^2}\,(2x+4M_\pi^2-t)\mathrm{Im}\,F^{(I_t=2)}(x,t)\ . \tag{36}$$

The integral (36), if convergent at all, converges even more slowly than the sum rule (35). The asymptotic behavior of $F^{(I_t=2)}$ is expected to be dominated by $s^{2\alpha_\rho-1}$ arising from the $\rho-\rho$ Regge cut [27]. Assuming the validity of Eqs. (35) and (36) at $t=0$, one can use them in (19) and then, from (27) express the two subtraction constants a and b:

$$\begin{aligned} b = \quad & \frac{1}{6\pi}\int_{4M_\pi^2}^{E^2}\frac{dx}{x^2}\mathrm{Im}\left\{2f_0^0(x)-5f_0^2(x)+\frac{9x}{x-4M_\pi^2}f_1^1(x)\right\} \\ & +\frac{1}{\pi}\int_{E^2}^{\infty}\frac{dx}{x(x-4M_\pi^2)}\mathrm{Im}\,F^{(I_t=1)}(x,0), \end{aligned} \tag{37}$$

$$\begin{aligned} (a+2b)M_\pi^2 = \quad & \frac{1}{2\pi}\int_{4M_\pi^2}^{E^2}\frac{dx}{x}\mathrm{Im}\left\{(1+\frac{2M_\pi^2}{x})[2f_0^0(x)+f_0^2(x)]-9\frac{x-2M_\pi^2}{x-4M_\pi^2}f_1^1(x)\right\} \\ & +\frac{3}{\pi}\int_{E^2}^{\infty}\frac{dx}{x^2}(x+2M_\pi^2)\mathrm{Im}\,F^{(I_t=2)}(x,0). \end{aligned} \tag{38}$$

These relations promote Eqs. (34) to two additional sum rules for α and β.

The slow convergence of the high-energy integrals in Eqs. (37) and (38) prohibits any practical use of these sum rules for α and β. The sum rule for β, which is expected to converge better, leads to values $\beta = 1.2-1.4$ which have the correct order of magnitude (as compared to the χPT prediction), but the uncertainty due to the high-energy tail of the integral in (37) is difficult to estimate. Needless to say, the evaluation of the sum rule for α which involves the integral (38) is even more problematic.

In the following, these barely convergent sum rules for β and α will be ignored. The above discussion should mainly serve as an explanation of why, in contrast to λ_1,...,λ_4, the two parameters α and β cannot be determined from the existing $\pi\pi$ scattering data at medium energies. New, high precision low-energy experiments are, indeed, unavoidable for this purpose.

3. Evaluation of the sum rules

In this section, the details of the evaluation of the four sum rules (33) are presented. The input data above 0.5 GeV and their use are discussed in subsection 3.1. The treatment of the low energy part is explained in subsection 3.2, where the final results are also presented. Subsection 3.3 is devoted to a careful analysis of error bars. Finally, subsections 3.4 and 3.5 contain a comparison with related works, in particular with the recently published standard χPT two loop calculation [19].

3.1 Energy region above 0.5 GeV

The main sources of experimental information concerning the $\pi\pi$ amplitude are the production experiments (see [25]), which give reliable results in the range $0.5 \le \sqrt{s} \le 1.9$ GeV. These,

together with (24), imply that the terms h_i and h_i', which are integrals over the range $[E^2, \infty]$, can essentially be evaluated using experimental data. The contribution from the asymptotic region $\sqrt{s} > 1.9$ GeV can be estimated using Regge phenomenology and it turns out that this contribution is fairly small. This is illustrated in table 1 below, which shows the contributions of the various energy ranges to these integrals.

$\sqrt{x}\,\mathrm{GeV}$	[0.5,0.95]	[0.95,1.9]	$[1.9, \infty]$	total
$h_0(\mathrm{GeV}^{-4})$	-6.15	-0.59	-0.08	-6.82
$h_2(\mathrm{GeV}^{-4})$	0.17	-0.15	0.00	0.02
$h_1(\mathrm{GeV}^{-6})$	-4.34	-0.11	-0.002	-4.45
$h_0'(\mathrm{GeV}^{-6})$	8.15	-0.54	-0.13	7.48
$h_2'(\mathrm{GeV}^{-6})$	11.82	-0.20	0.00	11.62

Table 1: *Contributions from the various energy ranges to the integrals h_i and h_i', with $E = 0.5$ GeV*

The table shows clearly that the integrals are rapidly convergent and are dominated by the region $\sqrt{s} < 1$ GeV with the exception, however, of h_2. In that case, the integrand is very small below 1 GeV because of cancellations among the three isospin contributions, and the whole integral is much suppressed as compared to the others. This is reminiscent of the approximation made in [28] of setting the combination of amplitudes with $I_t = 2$ equal to zero in sum rules derived there for λ_1, λ_2. One should be aware that this simple approximation breaks down for h_2'; as can be seen from the table, h_2' is even larger than h_0'.

We now specify in more detail how we have treated the experimental data in order to obtain the numbers given above and the central values for the λ_i's as given in the sequel. Let us first discuss the region below 1.9 GeV. The region was divided into two subregions which were treated somewhat differently:

$$a)\, 0.95 \leq \sqrt{s} \leq 1.9 \text{ GeV} \quad \text{and} \quad b)\, 0.5 \leq \sqrt{s} \leq 0.95 \text{ GeV}$$

- In the subregion a) we have employed the data obtained by Hyams *et al.* [29] (based on the CERN-Munich production experiment [30], which has by far the best statistics to date) in the form of the analytic K-matrix type parametrization which the authors provide [5]. Phase shifts and inelasticities have been determined for partial waves up to $\ell = 3$, which means that the contributions from the resonances $f_2(1270)$ and $\rho_3(1690)$ are automatically included. Partial waves with $\ell \geq 4$ were ignored: including the resonance $f_4(2040)$ in the narrow width approximation proved to have a negligibly small effect. In the case of the isospin $I = 2$ and the partial wave $\ell = 0$, we have used the data of Hoogland *et al.* [31] which have better statistics in this channel, and we have neglected the contributions of the higher partial waves.

[5] There are a few obvious misprints in the published formulae.

- In the subregion b), some care is needed because the results are very sensitive to the values of the phases there, particularly to the S and P partial waves. Concerning the $I = 0$ S-wave, several slightly different analyses of the CERN-Munich data have been performed. We shall use, as before, the data of Hyams *et al.* which extend down to 0.6 GeV, and also the results of Estabrooks and Martin [32] who provide the phases down to 0.5 GeV. We have performed a fit to both sets of data using the parametrization proposed by Schenk [33]:

$$\tan\delta_\ell^I(s) = p^{2\ell}\sqrt{1-\frac{4M_\pi^2}{s}}\left(\frac{4M_\pi^2 - s_0}{s - s_0}\right)(a_\ell^I + cp^2 + dp^4)\ , \qquad p^2 = \frac{s}{4M_\pi^2} - 1\ , \qquad (39)$$

  where c, d and s_0 are three parameters to be determined by the fit. In principle, one could take a_ℓ^I to be a free parameter as well, since we intend to use (39) rather far from the threshold, in the energy region from 0.5 to 1 GeV. However, we have preferred to keep with the idea of [33], taking a_ℓ^I to be the experimental value of the scattering length, in order not to give too much weight to the first few points around 0.5 GeV. The quality of the fit is practically independent of the exact value of a_ℓ^I, within the range allowed by experiment. The numerical values that we have used are collected in table 2. In the case of the P-wave, the data deduced from the CERN-Munich experiment are known to be in slight conflict with the results based on the Roy equations below 0.7 GeV [34, 35]. In that case, the Roy equation studies suggest that the δ_1^1 phase is very close to a pure Breit-Wigner shape below the resonance. Based on this prejudice, we have again used the parametrization (39), imposing the experimental value of the scattering length and adjusting the remaining three parameters in order to fit the experimental values of the ρ mass and of the ρ width, and such that $\delta_1^1(\sqrt{s} = 0.95\ \text{GeV})$ matches to the experimental value of [29]. If one uses a real fit to the data of [29] instead, then the difference in the results would be of the order of 5%, which means that they are perfectly compatible within the errors. Finally, for the $I = 2$ $\ell = 0$ partial wave we have fitted the parameters of the representation (39) to the data of [31]. In that case, we have used the same parametrization in both region a) and region b).

For comparison, we have also used the parametrizations of the $I = 0$ S-wave given by [36, 37], of the $I = 1$ P-wave given by [38] and the $I = 2$ S-wave given by [29]. Our final results are consistent with these fits to the data.

	a_ℓ^I	c	d	$\sqrt{s_0}\,(\text{MeV})$
I=0 $\ell = 0$	0.26	0.2535	-0.0200	843
I=2 $\ell = 0$	-0.028	-0.2325	-0.0160	0
I=1 $\ell = 1$	0.038	$0.2560\,10^{-3}$	$-0.6009\,10^{-4}$	769

Table 2: *Numerical values of the parameters used in (39) for the S and P partial waves. The values of the scattering lengths are taken from [39]*

In the asymptotic region we have assumed that the amplitudes are given by Regge phenomenology, and we have used the same parameters as in formulae (17), (18) and (19) of [40]. This representation was assumed to hold for $\sqrt{s} \geq 3$ GeV, and in the region between 1.9 and 3 GeV the amplitude was determined by linear interpolation.

3.2 Energy region below 0.5 GeV and results

We must now deal with the terms I_a^k in the sum rules. According to the definition (32), they are obtained by making a Taylor expansion of the difference $W_a^E(z) - \hat{W}_a(z)$. Here, $\hat{W}_a(z)$ is known analytically in terms of α, β and is a linear function of λ_1, λ_2. Since E is constrained not to exceed 1 GeV, one can ignore inelasticities to a good approximation and, using unitarity, express W_a^E in terms of phase shifts:

$$W_a^E(z) = \frac{s^2}{\pi} \int_{4M_\pi^2}^{E^2} dx \sqrt{\frac{x}{x-4M_\pi^2}} \frac{\sin^2 \delta_0^a(x)}{x^2(x-z)} \quad (a=0,2) \ ,$$
$$W_1^E(z) = \frac{s}{\pi} \int_{4M_\pi^2}^{E^2} dx \sqrt{\frac{x}{x-4M_\pi^2}} \frac{\sin^2 \delta_1^1(x)}{x(x-4M_\pi^2)(x-z)} \ . \tag{40}$$

In the upper part of the integration range, 0.5 GeV $\leq \sqrt{x} \leq E$, we can use the experimental phase shifts as discussed above. Below 0.5 GeV, experimental results for the phase shifts are rather limited. Fortunately, it is precisely in this range that we can make use of the chiral expansion of the amplitude. One expects that the chiral two-loop representation for the three phase shifts δ_0^0, δ_0^2 and δ_1^1 should be reasonably precise in the range $2M_\pi \leq \sqrt{s} \leq 500$ MeV: this expectation will be verified more quantitatively by comparing our results with those of a numerical solution of the Roy equations. We will therefore employ the chiral representation of the phase shifts in the dispersive integrals (40) below 0.5 GeV. When calculated in this way, the set of parameters I_a^k depend on α, β and also, in a nonlinear way, on the four λ_i's. A less precise procedure, but equally acceptable to the desired chiral order, is to use, below 500 MeV, the imaginary parts of the S and P partial wave amplitudes in the chiral two-loop approximation (which is known to violate unitarity appreciably close to 500 MeV); *i.e.*, one uses in the integrands of Eq.(15)

$$\mathrm{Im}\, f_0^a(x) = \sum_i w_a^i(x) \mathrm{Im}\, \bar{K}_i(x) \qquad (a=0,2)$$
$$\mathrm{Im}\, f_1^1(x) = (x-4M_\pi^2) \sum_i w_1^i(x)\, \mathrm{Im}\, \bar{K}_i(x) \ . \tag{41}$$

In this approximation, the dependence on λ_1, λ_2 is simply linear and there is no dependence upon λ_3 and λ_4. In general, the system of equations (33) must be solved in a self-consistent way.

Let us now discuss the numerical results. First, we consider the stability with respect to variation of the energy parameter E. Obviously, the parameters $\lambda_1, ..., \lambda_4$ should be independent of E. The validity of the formulas (33), as discussed in the preceeding section, requires that E

should be smaller than, roughly, 1 GeV (see the discussion preceeding (14)) and at the same time that $E^2 >> 4M_\pi^2$. These conditions suggest to consider an interval

$$500 \text{ MeV} < E < 1000 \text{ MeV} \ . \tag{42}$$

Variations of the results within this interval are shown in table 3 below [6]

E (MeV)	500	600	700	800	900
$10^3\lambda_1$	-4.86	-5.82	-6.04	-6.08	-6.11
$10^3\lambda_2$	9.68	9.64	9.60	9.56	9.55
$10^4\lambda_3$	3.33	2.55	2.31	2.20	2.16
$10^4\lambda_4$	-1.46	-1.49	-1.48	-1.46	-1.45

Table 3: *Results from the sum rules (33) for several values of the energy parameter E. The numbers correspond to $\alpha = 2$ and $\beta = 1.08$*

Clearly, one does not observe exact stability. The variation is more significant for λ_1 and λ_3 than for the remaining two parameters, which are stable within 5%. A reasonable stability plateau forms for all four parameters for values of E above $700 - 800$ MeV. Keeping in mind that E should remain sufficiently small such that the imaginary part of the D-wave contributions can be neglected, we shall assume in what follows that $E = 800$ MeV provides a reasonable compromise.

Let us now discuss the results for the λ_i's corresponding to various values of the two parameters α and β. This variation is of interest for the purpose of using the chiral formulas in a fit to determine α and β from experiment. Recall that α is expected to be close to 1 (within, say, 20%) according to standard chiral perturbation theory. In contrast, the generalized χPT can accomodate values of α as large as $\alpha = 4$. At present, the best K_{l4} data available [41] are compatible with a relatively wide range of values, $1 \lesssim \alpha \lesssim 3$. Some results, illustrating the dependence on α and β, are collected in table 4 below (the numbers shown correspond to $E = 800$ MeV).

α	β	$10^3\lambda_1$	$10^3\lambda_2$	$10^4\lambda_3$	$10^4\lambda_4$
1.04	1.08	-5.68	9.32	2.21	-1.48
2.00	1.00	-6.60	9.39	2.12	-1.33
2.00	1.08	-6.08	9.56	2.20	-1.46
2.00	1.13	-5.70	9.67	2.26	-1.54
2.50	1.08	-6.27	9.68	2.19	-1.45
3.00	1.08	-6.45	9.78	2.19	-1.44
3.50	1.08	-6.62	9.88	2.18	-1.44
4.00	1.08	-6.79	9.97	2.17	-1.43

[6] As in I, we use $F_\pi = 92.4$ MeV and $M_\pi = M_{\pi^+} = 139.57$ MeV.

Table 4: *Results for the parameters λ_1,...,λ_4 corresponding to several values of α and β.*

The first line in the table corresponds to the values of α and β obtained in the standard χPT at chiral order $O(p^4)$ in the three flavour case (see I). For the particular value $\alpha = 2$, we have varied β in the range allowed by the Morgan-Shaw band. The table shows that as long as α and β remain in the range allowed by experiment, the variation of the values of the λ_i's remains smaller than the error bars (see table 6). As a first approximation one may ignore this variation and adopt as average values

$$10^3\lambda_1 \simeq -6.1, \quad 10^3\lambda_2 \simeq 9.6, \quad 10^4\lambda_3 \simeq 2.2, \quad 10^4\lambda_4 \simeq -1.45 \; . \tag{43}$$

These numbers update those given in our previous paper I

$$10^3\lambda_1 \simeq -5.3, \quad 10^3\lambda_2 \simeq 9.7, \quad 10^4\lambda_3 \simeq 2.9, \quad 10^4\lambda_4 \simeq -1.40 \; , \tag{44}$$

which were computed using a smaller value of E ($E = 550$ MeV), and with a slightly different treatment of the experimental data. The results are compatible within the uncertainties and the differences do not affect the numbers given in I for the threshold parameters. The results at higher energies are obviously more affected by small modifications of the values of λ_1,...λ_4. For instance, computing the value of the phase of the ϵ' parameter with the numbers from table 4, one would find a value higher by $2-3^\circ$ than the one given in I.

Once the λ_i's have been determined self-consistently, one can return to the sum rule expressions and examine the contribution of each integration region. These are displayed in table 5 which shows that the contribution from the resonance region, 0.5 GeV to 1 GeV, dominates the values of the three parameters λ_1, λ_2 and λ_3, while the contribution from the low energy region is smaller though not negligible. The situation is different in the case of λ_4, for which the low energy region dominates. One may then fear that λ_4 could be overly sensitive to higher order chiral corrections. However, this does not seem to be the case: an estimate of the $O(p^8)$ uncertainties is performed in the next subsection which shows that λ_4 is weakly affected (see table 6).

$\sqrt{x}$ (GeV)	[$2M_\pi$,0.5]	[0.5,1]	[1,1.9]	[1.9,∞]
$10^3\,\lambda_1$	2.72	-8.84	0.14	-0.10
$10^3\,\lambda_2$	1.35	7.69	0.42	0.10
$10^4\,\lambda_3$	0.49	1.68	0.03	-0.001
$10^4\,\lambda_4$	-1.15	-0.28	-0.02	-0.001

Table 5: *Contributions of four successive integration ranges to the values of λ_1,...λ_4 with $\alpha = 2$, $\beta = 1.08$.*

3.3 Discussion of the errors

We can identify two sources of uncertainties in the estimate of the λ_i's: i) the errors affecting the experimental phase shifts and ii) the errors affecting the theoretical phase shifts, *i.e.* contributions of chiral order $O(p^8)$ or higher that one might expect to become sizable in the neighborhood of 500 MeV. In order to estimate the uncertainties arising from the former source, we proceed as follows. First we take E to be $E = 0.5$ GeV. Using this value for E has the advantage that the experimental phases show up only in the integrals h_i, h_i' and not in the terms I_a^k. In computing I_a^k, we may use the approximation (41) so that the sum rules (33) can be solved analytically (since they reduce to a linear system) for λ_1,...,λ_4. Finally, we assume that the errors on the phase shifts are piecewise constant as a function of energy (this assumption is seriously violated only in a narrow region around the $K\bar{K}$ threshold), and we have adopted the following values, an educated guess inspired by [29, 32, 31]:

$$\begin{array}{llll} 0.50 \leq \sqrt{s} \leq 0.95\text{GeV} & \Delta\delta_0^0 = 4^\circ & \Delta\delta_1^1 = 1^\circ & \Delta\delta_0^2 = 2^\circ \\ 0.95 \leq \sqrt{s} \leq 1.90 & \Delta\delta_0^0 = 9^\circ & \Delta\delta_1^1 = 1^\circ & \Delta\delta_0^2 = 4^\circ \ . \end{array} \tag{45}$$

Under these assumptions the errors are given as simple analytic expressions in terms of a few phase shift integrals and it is a simple matter to calculate them. With the choice $E = 0.5$ GeV, table 1 shows that most of the contributions to the integrals h_i are concentrated in the integration region below 1 GeV. We have therefore neglected the errors coming from either the inelasticities or the partial waves with $\ell \geq 2$, or from the asymptotic contributions. Finally, in order to estimate the error coming from the missing higher chiral orders in the low energy integrals, we have compared the result of calculating $W_a^E(z)$ using three evaluations of the integrands which have identical chiral expansions up to (and including) chiral order six, and which differ at higher chiral order: i) we use expressions (40) together with the chiral approximation to the phase shifts (see formula (4.13) of I) ii) we use the same expression but get the phase shift from a Padé approximant to the pertubative expansion which satisfies unitarity exactly (this generalizes to the $O(p^6)$ case [7] the idea used in the context of χPT in [42]) and iii) we use the chiral approximation (41), which violates unitarity. We have collected the results of these error estimates in table 6 where we show separately the contribution from each source, as discussed above.

	$\Delta\delta_0^0$	$\Delta\delta_1^1$	$\Delta\delta_0^2$	$O(p^8)$	Total
$10^3\Delta\lambda_1$	1.02	1.47	0.40	0.40	2.23
$10^3\Delta\lambda_2$	0.05	0.26	0.18	0.20	0.52
$10^4\Delta\lambda_3$	0.34	0.25	0.14	0.20	0.64
$10^4\Delta\lambda_4$	0.01	0.03	0.09	0.02	0.12

[7] For the S and P partial wave amplitudes, the chiral expansion consists in three successive terms $f = f^{(2)} + f^{(4)} + f^{(6)}$ and one finds $f^{Padé} = f^{(2)}/\big[1 - f^{(4)}/f^{(2)} + (f^{(4)}/f^{(2)})^2 - f^{(6)}/f^{(2)}\big]$.

Table 6: *Errors on λ_1,...,λ_4 arising respectively from the experimental errors on δ_0^0, δ_1^1 and δ_0^2 and from $O(p^8)$ corrections to the amplitude below 0.5 GeV.*

The entries in the table were evaluated for $\alpha = 2$, $\beta = 1.08$, but the errors show no significant variation with α or β. We have added the errors associated with the experimental errors on the three phase shifts in quadrature, as it seems reasonable to assume that they are independent. The last piece of the error was added linearly. The reason why λ_2 and λ_4 have a small relative error compared to λ_1 and λ_3 is that the contribution from the isospin $I = 0$ amplitude essentially drops out. (There is only an indirect small contribution which comes from the fact that the determination of λ_2 and λ_3 depends on λ_1). The values of the parameters λ_2 and λ_4 are thus to a large extent controlled by the ρ resonance.

3.4 Comparison with former results

The parameters λ_3, λ_4, being of chiral order $O(p^6)$, have not been discussed previously in the literature. At $O(p^4)$, a potentially accurate determination of the parameters λ_1 and λ_2 is possible, making use of the linear relationship with the low energy constants L_1, L_2 and L_3 (see I: the relation is the same in the standard and in the generalized χPT). These three constants can be separately determined from the form factors of the K_{l4} decay amplitude. On the theoretical side, this amplitude was analyzed at order one loop of the χPT [43, 44] and estimates of higher order corrections were made [45]. New high statistics data on K_{l4} decays would enable a fairly accurate determination of these constants. The best results available at present [45], which make use of the experimental results of Rosselet *et al.* [41] together with the values of the $\pi\pi$ D-wave scattering lengths quoted in [39], lead to

$$\lambda_1 = (-6.4 \pm 6.8)\, 10^{-3} \ , \qquad \lambda_2 = (10.8 \pm 1.2)\, 10^{-3} \quad \text{(one loop)} \ . \tag{46}$$

If one uses solely the constraints from the D-wave scattering lengths, then, according to [2, 45], the error on λ_1 would be increased by 50% and the error on λ_2 would be three times larger, while the central values would remain approximately the same. In comparing these central values with the numbers that we quote in table 3, one must keep in mind that in the latter λ_1 and λ_2 include corrections of chiral order six. As far as the order of magnitude of these corrections is concerned, a reasonable guess should be provided by the values of λ_3 or λ_4. This leads one to expect that these corrections should not exceed $2 - 3\%$. On the contrary, we find that the $O(p^6)$ corrections to the D-wave scattering lengths are significant. This is illustrated in table 7, where the values of a few threshold expansion parameters which pick up their leading contribution at $O(p^4)$ are shown, both at $O(p^4)$ and at $O(p^6)$ (the values of α, β, λ_1,...λ_4 being kept the same).

These contributions explain why our central value for λ_2 is smaller than (46) by roughly 10%. Apart from this effect, we believe that the errors on the LEC's (low energy coefficients) extracted from these D-wave scattering lengths have been somewhat overestimated by treating

them as independent experimental data. In reality, the numbers quoted in [39] are obtained as sum rules from the Roy dispersive representation (30) projected on $\ell = 2$, and are evaluated using experimental data at high energy and extrapolation of these data down to the threshold, based on the numerical solutions to the Roy equations of [23]. It is obviously more efficient, as far as errors are concerned, to express directly the LEC's as sum rules. Concerning the P-wave, the results found in the chiral expansion for the threshold parameters a_1^1, b_1^1 and c_1^1 tend to support the idea that the P-wave has a nearly pure Breit-Wigner shape down to the threshold. This indeed implies the following relations:

$$a_1^1 = \frac{4\Gamma_V M_V^2}{(M_V^2-4)^{\frac{5}{2}}}, \quad b_1^1 = \frac{4a_1^1}{(M_V^2-4)}, \quad c_1^1 = \frac{4b_1^1}{(M_V^2-4)}\ , \tag{47}$$

where the mass and the width are expressed in units of the pion mass. One can check that these relations are rather well satisfied for a_1^1 and b_1^1 and remain qualitatively correct even for c_1^1. An evaluation of b_1^1 on the basis of sum rules was performed only recently as an outcome of new, rapidly convergent sum rules involving combinations of threshold parameters [46] and the result is $b_1^1 = (6 \pm 4) \times 10^{-3}$.

	1-loop	1-loop+2-loop	Ref[39]
$10^3\, a_2^0$	1.52	1.72	1.7 ± 0.3
$10^3\, a_2^2$	0.20	0.14	0.13 ± 0.3
$10^3\, b_1^1$	4.09	5.46	–
$10^4\, a_3^1$	0.30	0.58	0.6 ± 0.2
$10^4\, b_2^0$	−4.79	−3.41	–
$10^4\, b_2^2$	−3.04	−3.54	–
$10^4\, c_1^1$	−1.82	4.64	–

Table 7: *Values of a few threshold parameters at one-loop and at two-loop accuracy (in units of M_{π^+}) for $\alpha = 2$ and $\beta = 1.08$. The threshold parameters in the last four lines have no tree-level contributions at order $O(p^4)$.*

It is instructive to compare our results for the S and P partial-wave amplitudes derived in the two-loop approximation with a direct numerical solution of the Roy equations. This comparison allows to gauge the importance of $O(p^8)$ terms which are present in the numerical solution and also allows to verify whether the driving terms agree. Numerical results for the phase shifts have been tabulated by Froggatt and Petersen [40] corresponding to $a_0^0 = 0.30$ and $a_0^2 = -0.018$. Using the chiral expansion of the low-energy parameters (see appendix D of I) and the sum rules, we obtain the following numbers for the chiral parameters corresponding to these scattering lengths: $\alpha = 2.84$, $\beta = 1.09$, $10^3\lambda_1 = -6.32$, $10^3\lambda_2 = 9.77$, $10^4\lambda_3 = 2.20$ and $10^4\lambda_4 = -1.46$. Figure 1 shows that the phase shifts computed from the two-loop chiral expansion agree rather

well with the results of [40] up to an energy $\sqrt{s} \simeq 500$ MeV, thereby justifying the assumption made in the derivation of the sum rules to trust the chiral representation precisely in this range.

3.5 Implications for the standard χPT two-loop amplitude parameters

While the present article was being completed, the computation of the $\pi\pi$ scattering amplitude to two-loop accuracy has been achieved in the framework of the standard χPT in Ref.[19]. Comparing the expression of $A(s|t,u)$ obtained by the authors of [19] with the one we had derived in I, we conclude that they coincide up to order $O(p^8)$ contributions, provided one identifies the six constants b_i which appear in the amplitude of [19] with our parameters $\alpha,\ \beta,\ \lambda_i$ as follows [8]:

$$
\begin{aligned}
b_3 &= \lambda_1 + \frac{1}{2}\lambda_2 - \frac{M_\pi^2}{F_\pi^2}\left(6\lambda_3 + \frac{1}{12288\pi^2} - \frac{103}{4608\pi^4}\right)\\
b_4 &= \frac{1}{2}\lambda_2 + \frac{M_\pi^2}{1536\pi^4 F_\pi^2}\\
b_5 &= \lambda_3 - \frac{1}{4}\lambda_4 - \frac{19}{1536\pi^4}\\
b_6 &= -\frac{3}{4}\lambda_4 - \frac{7}{4608\pi^4}\\
b_1 &= \frac{F_\pi^2}{3M_\pi^2}(\alpha-1) - \frac{4}{3}\frac{F_\pi^2}{M_\pi^2}(\beta-1) + 4\lambda_1 - \frac{M_\pi^2}{F_\pi^2}\left(8\lambda_3 + \frac{1}{1152\pi^2} - \frac{1}{144\pi^4}\right)\\
b_2 &= \frac{F_\pi^2}{M_\pi^2}(\beta-1) - 4\lambda_1 + \frac{M_\pi^2}{F_\pi^2}\left(12\lambda_3\ + \frac{19}{18432\pi^2} - \frac{1}{96\pi^4}\right).
\end{aligned}
\tag{48}
$$

From the last two formulae, it is apparent that in the standard framework, both α and β have to become equal to one in the chiral limit. (In $G\chi$PT, α stays away from 1 and b_1 becomes large, signaling the breakdown of the standard expansion.)

From our preceding determination of the four constants λ_i, we can now deduce the values of the four constants $b_3,\ b_4,\ b_5$ and b_6. In order to make the result as accurate as possible, the $O(p^6)$ values of α and β in the standard framework would be required. Unfortunately, we see no way to extract these values from the content of [19]. However, we expect that α and β will remain close to their leading order values $\alpha = \beta = 1$ in the standard framework. The $O(p^4)$ values (in the two flavour standard χPT) $\alpha = 1.06$, $\beta = 1.10$ provide already some information about the expected corrections. Using these values, we obtain (taking $F_\pi = 93.2$ MeV here in order to conform with [19]):

$$
\begin{aligned}
b_3 &= (-3.7 \pm 2.4)\,10^{-3} \qquad & b_4 &= (4.8 \pm 0.3)\,10^{-3}\\
b_5 &= (1.4 \pm 0.6)\,10^{-4} \qquad & b_6 &= (1.0 \pm 0.1)\,10^{-4}
\end{aligned}
\tag{49}
$$

If one assumes that the $O(p^6)$ corrections to α and β are of the order of the square of the $O(p^4)$ corrections, then the above numbers are not modified.

[8] We are indebted to G. Colangelo for pointing out a mistake in the first version of the manuscript.

4. Summary and conclusions

i) Among the six parameters $\alpha,\ \beta,\ \lambda_1, ..., \lambda_4$ that define the two-loop $\pi\pi$ scattering amplitude, the parameter α is the most sensitive to the chiral structure of the QCD vacuum. Furthermore, it is relatively weakly affected by higher orders of χPT, less affected than, for instance, the S-wave scattering lengths. For these reasons, the parameter α represents a suitable quantitative characteristic of the strength of quark condensation, and it should be determined experimentally in order to disentangle the large and small condensate alternatives. To accomplish this, one needs new high-precision low-energy $\pi\pi$ scattering data *and* a model-independent determination of the parameters $\lambda_1, ..., \lambda_4$. Indeed, this would allow a measurement of α (and β) from a simultaneous fit to several low-energy observables such as $\delta_0^0 - \delta_1^1$ [20] and $a_0^0 - a_0^2$ [21]. The present paper contributes to this program by providing a rather accurate determination of the parameters $\lambda_1, ..., \lambda_4$ using existing $\pi\pi$ scattering data at medium energies.

ii) The method of determination follows from a rather systematic procedure: first, one establishes a dispersive representation for the low-energy amplitude in terms of absorptive parts and two unknown subtraction *constants*. This representation is displayed in Eq.(30), and it holds up to corrections of order $O[(p/E)^8]$, with $4M_\pi^2 << E^2 \lesssim 1$ GeV^2. Its derivation parallels the derivation of the Roy equations, using twice-subtracted fixed-t dispersion relations and exploiting crossing symmetry. The second step consists in identifying, up to and including $O(p^6)$ accuracy, the dispersive representation (30) and the perturbative two-loop amplitude in a whole low-energy domain of the Mandelstam plane. The comparison yields six, and only six, sum rules for $\alpha,\ \beta,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$. The sum rules for α and β are only barely convergent and they were ignored. One remains with four rapidly convergent sum rules for $\lambda_1, ..., \lambda_4$. Our method minimizes the error in the determination of these parameters, since it makes full use of the information contained in crossing symmetry.

iii) In evaluating the sum rules (33), one uses the existing $\pi\pi$ data in the energy range $0.5\ \text{GeV} < \sqrt{s} < 1.9\ \text{GeV}$ which were extracted from unpolarized high statistics $\pi N \to \pi\pi N$ high-energy production experiments. The outcome is rather sensitive to these data; in particular, their error bars constitute the main source of uncertainty in the values of the λ_i's (see table 6). Our machinery is ready to accept as input any other set of medium energy $\pi\pi$ phases and inelasticities, provided they are consistent with the Roy-type dispersion relations. (The standard $\pi\pi$ phases extracted from the old Cern-Munich experiment [30] have been recently criticized [47]. However, no alternative phases have been proposed.) The high-energy tails in the sum rules (33) are estimated using the Regge-pole model and they are found to be negligible. Finally, the contribution of the low-energy range $2M_\pi \leq \sqrt{s} \leq 0.5$ GeV is evaluated using the perturbative two-loop amplitude itself. This contribution introduces a weak dependence of the parameters $\lambda_1, ...\lambda_4$ on α and β illustrated in table 4. This low-energy part of the sum rules encompasses the bulk of infrared contributions represented by the chiral logarithms in an explicit χPT calculation.

iv) Our final result can be read off from tables 3, 4 and 6. The variation of the λ_i's with the

cutoff E exhibits a plateau for $E \sim 700 - 900$ MeV. Choosing for the central values $E = 800$ MeV as well as the center of the variations with respect to α and to β, we obtain as the final result of this paper

$$10^3\lambda_1 = -6.1 \pm 2.2 \ , \qquad 10^3\lambda_2 = 9.6 \pm 0.5$$
$$10^4\lambda_3 = 2.2 \pm 0.6 \ , \qquad 10^4\lambda_4 = -1.45 \pm 0.12 \ . \tag{50}$$

The central values may be further refined taking into account the small dependence on α and on β. In any case, the variation with α and β is smaller than the error bars displayed in (50). Giving to the parameters λ_1,...,λ_4 the values (50), the two-loop perturbative amplitude becomes a faithful *analytic* low-energy representation of the numerical solution of the Roy equations, which were used to establish the values, usually quoted as "experimental", of $\pi\pi$ threshold parameters [39]. Indeed, for each pair of values of α and β (a_0^0 and a_0^2) the perturbative amplitude reproduces the S-wave slopes, and the P, D and F-wave threshold parameters as quoted in [39].

v) The parameters $\lambda_1, ..., \lambda_4$ are determined globally, without any reference to their expansion in powers of quark masses and chiral logarithms. Consequently, the resulting values (50) concern both the standard and the generalized χPT (modulo a small variation with α and β displayed in table 4). It would be interesting to see, whether the "chiral anatomy" of these parameters resulting from the recent standard χPT two-loop calculation [19] can be used to predict the values of $\lambda_1, ..., \lambda_4$ (or, equivalently, $b_3, ..., b_6$) in agreement with their present determination.

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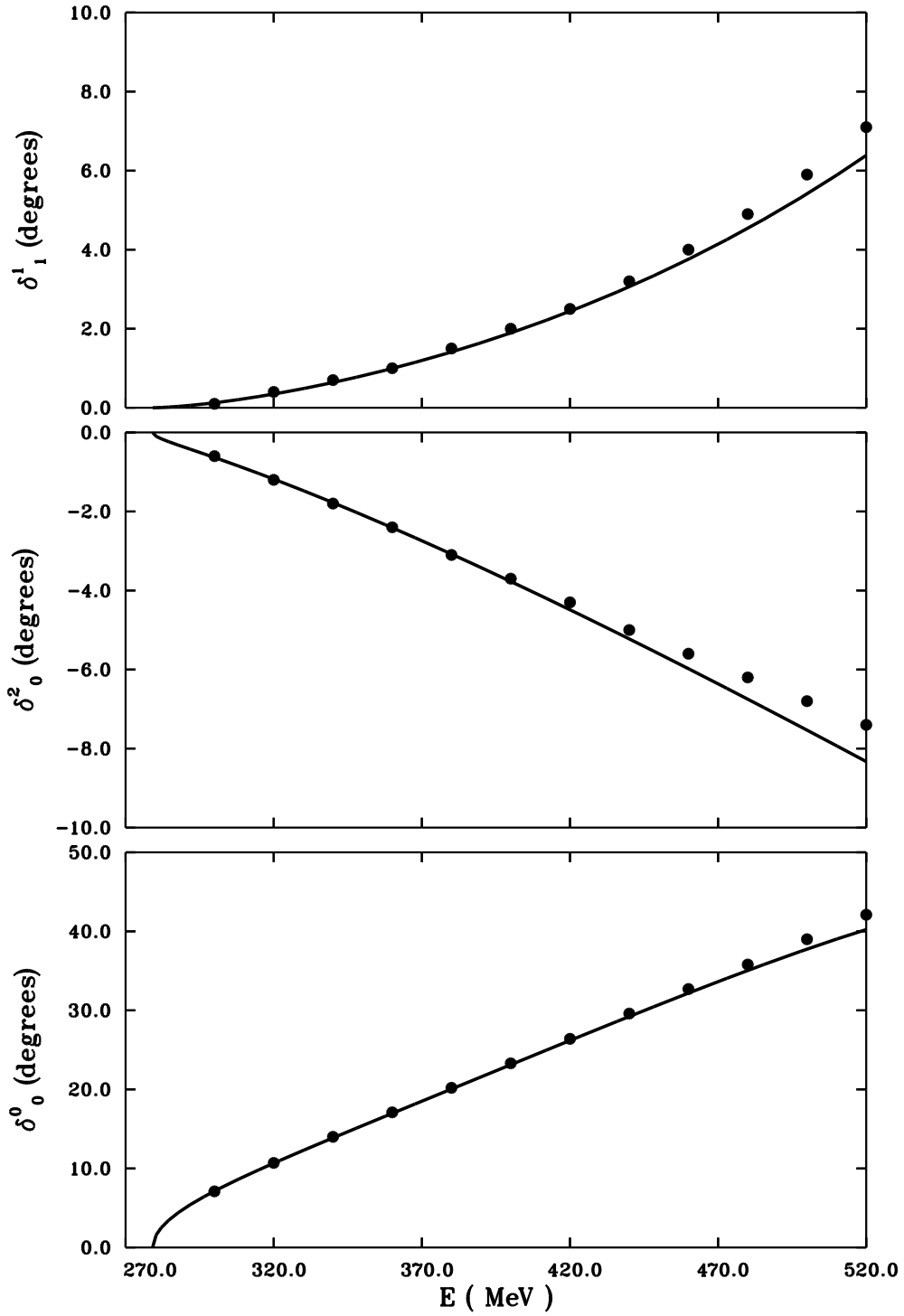

Figure 1: Comparison of the S- and P- wave phase shifts in the chiral expansion at two-loop with the results of a numerical solution of the Roy equations quoted by Froggatt and Petersen [40]